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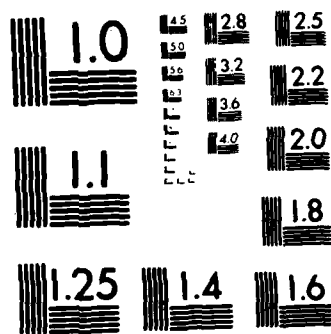
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ORDERING DISTRIBUTIONS BY  
SCALED ORDER STATISTICS

by

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# Abstract

Motivated by applications in reliability theory, we define a preordering  $(X_1, \dots, X_n) \stackrel{(k)}{\sim} (Y_1, \dots, Y_n)$  of nonnegative random vectors by requiring the  $k$ -th order statistic of  $a_1 X_1, \dots, a_n X_n$  to be stochastically smaller than the  $k$ -th order statistic of  $a_1 Y_1, \dots, a_n Y_n$  for all choices of  $a_i > 0, i = 1, 2, \dots, n$ . We identify a class of functions  $M_{k,n}$  such that  $X \stackrel{(k)}{\sim} Y$  if and only if  $E\phi(X) \leq E\phi(Y)$  for all  $\phi \in M_{k,n}$ . Some preservation results related to the ordering  $\stackrel{(k)}{\sim}$  are obtained. Some examples and applications of the results are given.

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## 1. Introduction.

Consider the set  $\mathfrak{X}_n$  of all  $n$ -dimensional random vectors which are nonnegative with probability one. For  $\underline{z} = (z_1, \dots, z_n) \in R_+^n \equiv \{\underline{z} : \underline{z} \geq \underline{0}\}$ , denote by  $\underline{z}_{(k)} = (z_1, \dots, z_n)_{(k)}$  the  $k$ -th smallest  $z_i$  in  $\{z_1, \dots, z_n\}$ . Thus for  $\underline{Z} = (Z_1, \dots, Z_n) \in \mathfrak{X}_n$ , the  $k$ -th order statistic of  $Z_1, \dots, Z_n$  is  $\underline{Z}_{(k)} = (Z_1, \dots, Z_n)_{(k)}$ .

It is possible to introduce various orderings on  $\mathfrak{X}_n$ . In this paper we consider the preordering  $\cdot \underset{\sim}{\overset{(k)}{\leq}}$  defined for  $\underline{X}, \underline{Y} \in \mathfrak{X}_n$  by

$$\underline{X} \underset{\sim}{\overset{(k)}{\leq}} \underline{Y}$$

if and only if for all  $a_i > 0$ ,  $i = 1, \dots, n$ ,

$$(a_1 X_1, \dots, a_n X_n)_{(k)} \overset{st}{\leq} (a_1 Y_1, \dots, a_n Y_n)_{(k)}$$

where here  $\overset{st}{\leq}$  denotes the usual (univariate) stochastic ordering. (For univariate random variables  $X$  and  $Y$  the notation  $X \overset{st}{\leq} Y$  means

$Eg(X) \leq Eg(Y)$  for all nondecreasing Borel measurable functions  $g$  for which the expectations exist.)

The following two results show the relationship between the preordering  $\underset{\sim}{\overset{(k)}{\leq}}$  and other well known orderings (see also Scarsini (1985)).

Proposition 1.1. Let  $\underline{X}$  and  $\underline{Y}$  be members of  $\mathfrak{X}_n$  with distributions  $F$  and  $G$  respectively. Then the following two conditions are equivalent:

- (i)  $\underline{X} \underset{\sim}{\overset{(n)}{\leq}} \underline{Y}$ ,
- (ii)  $F(\underline{t}) \geq G(\underline{t})$  for all  $\underline{t} \in R_+^n$ .

Proof: Clearly (i) is equivalent to

$$\max(a_1 X_1, \dots, a_n X_n) \overset{st}{\leq} \max(a_1 Y_1, \dots, a_n Y_n)$$

whenever  $a_i > 0$ ,  $i = 1, \dots, n$ . The latter is the same as

$$F\left(\frac{t}{a_1}, \dots, \frac{t}{a_n}\right) \geq G\left(\frac{t}{a_1}, \dots, \frac{t}{a_n}\right) \text{ whenever } t \geq 0, a_i > 0, i = 1, \dots, n,$$

which is the same as

$$(1.1) \quad F(t_1, \dots, t_n) > G(t_1, \dots, t_n) \text{ whenever } t_i > 0, i = 1, \dots, n.$$

Finally, using standard limiting arguments, (1.1) is equivalent to (ii). ||

Similarly one can prove:

Proposition 1.2. Let  $\underline{X}$  and  $\underline{Y}$  be members of  $\mathcal{K}_n$  with distributions  $F$  and  $G$  respectively. Then the following two conditions are equivalent:

$$(iii) \quad \underline{X} \stackrel{(1)}{\sim} \underline{Y},$$

$$(iv) \quad \bar{F}(\underline{t}) < \bar{G}(\underline{t}) \text{ for all } \underline{t} \in R_+^n,$$

where  $\bar{F}(t_1, \dots, t_n) \equiv P(X_1 > t_1, \dots, X_n > t_n)$  and

$\bar{G}(t_1, \dots, t_n) \equiv P(Y_1 > t_1, \dots, Y_n > t_n)$  are the corresponding survival functions.

Thus the ordering  $\stackrel{(1)}{\sim}$  is the same as the one discussed by Rüschendorf (1980) and Mosler (1984). Mosler (1984) also discusses the ordering described in (ii) which is equivalent to  $\stackrel{(n)}{\sim}$  as is shown in Proposition 1.1.

The purpose of this paper is to study the preordering  $\stackrel{(k)}{\sim}$  for  $k = 1, 2, \dots, n$ . We state the main results in Section 2. Some applications and examples are given in Section 3.

## 2. The main results.

For  $m \in \{1, \dots, n\}$  let  $\psi_m$  be the set of all subsets of  $\{1, \dots, n\}$  of size  $m$ . For  $I = \{i_1, \dots, i_m\} \in \psi_m$  and  $x_{i_1}, \dots, x_{i_m} > 0$ , let  $\underline{x}_I$  denote  $(x_{i_1}, \dots, x_{i_m})$  and let  $(\underline{x}_I, \infty)$  denote  $\lim_{\substack{x \\ \underline{x}_I c \rightarrow \infty}} (x_1, \dots, x_n)$  and  $(\underline{x}_I, \underline{0})$  denote  $\lim_{\substack{x \\ \underline{x}_I c \rightarrow \underline{0}}} (x_1, \dots, x_n)$ . In this paper  $I^c$  denotes the complement of  $I$  in  $\{1, \dots, n\}$ . Also,  $\infty$  denotes a single  $\infty$  or a vector  $(\infty, \dots, \infty)$  of a proper size. Similarly,  $\underline{0}$  denotes a vector of zeroes of a proper size.

Let  $M_1$  denote the class of all bounded distribution functions on  $R_+^n$

with no singularities at  $\infty$ , that is,  $f \in M_1$ , if and only if there exists a measure  $\mu$  on  $R_+^n$  such that  $f(t) = \mu([0, t])$  and for some bound  $L < \infty$  we have  $f(t) \rightarrow L$  as  $t \rightarrow \infty$ . Note that in this paper,  $\mu$  is allowed to have positive mass at the origin or along any of the axis or on sets of the form  $\{(\underline{x}_I, 0) : \underline{x} \in R_+^m\}$  for some  $I \in \psi_m$ . Clearly every  $f \in M_1$  determines a unique measure  $\mu$  as described above and vice versa. In the sequel, for

$\underline{x} \in R_+^n$ ,  $I \in \psi_m$  and  $f \in M_1$ , we denote  $\lim_{\substack{\underline{x}_I \rightarrow \infty \\ I^c \rightarrow 0}} f(x_1, \dots, x_n)$  by  $f(\underline{x}_I, \infty)$

and  $\lim_{\substack{\underline{x}_I \rightarrow 0 \\ I^c \rightarrow \infty}} f(x_1, \dots, x_n)$  by  $f(\underline{x}_I, 0)$ .

For  $k \in \{1, \dots, n\}$  let  $M_{k,n}$  be the class of functions  $\phi: R_+^n \rightarrow R$  such that

$$(2.1) \quad \phi(x_1, \dots, x_n) = \sum_{m=n-k+1}^n (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I \in \psi_m} f(\underline{x}_I, \infty), \text{ for some } f \in M_1,$$

where  $\sum_{I \in \psi_m}$  denotes the sum over all  $\binom{n}{m}$  elements of  $\psi_m$ . We sometimes suppress the subscript  $n$  of  $M_{k,n}$  and just write  $M_k$ . Note that for  $k = 1$ , the above definition and the previous definition of  $M_1$  coincide.

Theorem 2.1. Let  $\underline{x}, \underline{y} \in \mathcal{X}_n$ . Then the following two conditions are equivalent:

(v)  $\underline{x} \stackrel{(k)}{\sim} \underline{y}$ .

(vi)  $E\phi(\underline{x}) \leq E\phi(\underline{y})$  for all  $\phi \in M_k$  such that the expectations exist.

Proof. Suppose  $\underline{x} \stackrel{(k)}{\sim} \underline{y}$ . Let  $a_i > 0$ ,  $i = 1, \dots, n$ . From David (1970), p. 75 it follows that for every  $x > 0$ ,

$$\begin{aligned} & P\{(a_1 X_1, \dots, a_n X_n)_{(k)} > x\} \\ &= \sum_{m=n-k+1}^n (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{\{i_1, \dots, i_m\} \in \psi_m} P\{a_{i_1} X_{i_1} > x, \dots, a_{i_m} X_{i_m} > x\}. \end{aligned}$$



Thus

$$\begin{aligned} & P\{(a_1 X_1, \dots, a_n X_n)_{(k)} > x\} \\ &= E\left[\sum_{m=n-k+1}^n (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I \in \psi_m} x_{[(t_I, 0), \infty]}(\underline{x})\right] \end{aligned}$$

where  $t_i \equiv x/a_i$ ,  $i = 1, \dots, n$  and  $x_A$  is the indicator function of the set  $A$ . Now, if  $\underline{x} \stackrel{(k)}{\sim} \underline{y}$  and  $F$  and  $G$  are the distribution functions of  $\underline{x}$  and  $\underline{y}$  respectively, then

$$\begin{aligned} (2.2) \quad & \int_{R_+^n} \left[ \sum_{m=n-k+1}^n (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I \in \psi_m} x_{[(t_I, 0), \infty]}(\underline{x}) \right] dF(\underline{x}) \\ & \leq \int_{R_+^n} \left[ \sum_{m=n-k+1}^n (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I \in \psi_m} x_{[(t_I, 0), \infty]}(\underline{x}) \right] dG(\underline{x}). \end{aligned}$$

Let  $\phi \in M_k$  and let  $f$  be the corresponding member of  $M_1$  as described in (2.1). Then

$$\begin{aligned} E\phi(\underline{x}) &= \int_{R_+^n} \sum_{m=n-k+1}^n (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I \in \psi_m} f(\underline{x}_I, \infty) dF(\underline{x}) \\ &= \int_{R_+^n} \sum_{m=n-k+1}^n (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I \in \psi_m} \left[ \int_{R_+^n} x_{[0, (\underline{x}_I, \infty)]}(\underline{u}) df(\underline{u}) \right] dF(\underline{x}) \\ &= \int_{R_+^n} \left[ \int_{R_+^n} \sum_{m=n-k+1}^n (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I \in \psi_m} x_{[(\underline{u}_I, 0), \infty]}(\underline{x}) dF(\underline{x}) \right] df(\underline{u}) \\ (2.2) \quad & \leq \int_{R_+^n} \left[ \int_{R_+^n} \sum_{m=n-k+1}^n (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I \in \psi_m} x_{[(\underline{u}_I, 0), \infty]}(\underline{x}) dG(\underline{x}) \right] df(\underline{u}) \\ &= E\phi(\underline{y}) \end{aligned}$$

and (vi) follows.

Clearly  $\sum_{m=n-k+1}^n (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I \in \psi_m} x_{[(t_I, 0), \infty]}(\cdot)$  is a function in  $M_k$

whenever  $t > 0$ . It follows that (vi) implies (2.2) which is equivalent to (v). ||

Note that with  $k = 1$ , Theorem 2.1 yields Theorem 3(a) of Rüschendorf (1980) for measures on  $R_+^n$ .

The following preservation results will be used in Section 3.

**Theorem 2.2.** Assume that  $\underline{X} \stackrel{(k)}{\sim} \underline{Y}$  for some  $k \in \{1, \dots, n\}$  and  $\underline{X}$  and  $\underline{Y}$  in  $\mathcal{X}_n$ . Let  $b_i: R_+ \rightarrow R_+$  be a right continuous nondecreasing function,  $i = 1, \dots, n$ . Then

$$(2.3) \quad (b_1(X_1), \dots, b_n(X_n)) \stackrel{(k)}{\sim} (b_1(Y_1), \dots, b_n(Y_n)).$$

Theorem 2.2 can be proved by an explicit computation of

$$P\{(a_1 b_1(X_1), \dots, a_n b_n(X_n))_{(k)} > t\} \text{ and}$$

$P\{a_1 b_1(Y_1), \dots, a_n b_n(Y_n))_{(k)} > t\}$ . However we note that it is also an immediate consequence of Theorem 2.1 and the following Lemma 2.3. Lemma 2.3 is also used in the proof of Theorem 2.4. The proof of Lemma 2.3 is easy and is omitted.

**Lemma 2.3.** If  $\phi(\cdot, \dots, \cdot) \in M_{k,n}$  and if  $b_i: R_+ \rightarrow R_+$  is right continuous nondecreasing function,  $i = 1, \dots, n$ , then the function defined by

$$\phi(b_1(\cdot), \dots, b_n(\cdot)) \text{ is also a member of } M_k.$$

The following result is important because it yields Theorems 2.6 and 2.7 below as special cases.

**Theorem 2.4.** Let  $\underline{X}$ ,  $\underline{Y}$ ,  $\underline{Z}$  and  $\underline{W}$  be random vectors in  $\mathcal{X}_n$  such that

$$\begin{aligned} \underline{X} &\stackrel{(k)}{\sim} \underline{Y}, \\ \underline{Z} &\stackrel{(k)}{\sim} \underline{W}, \end{aligned}$$

and  $\underline{X}$  and  $\underline{Z}$  are independent and  $\underline{Y}$  and  $\underline{W}$  are independent. Then

$$\underline{S} \equiv (c_1(x_1, z_1), \dots, c_n(x_n, z_n))$$

$$\stackrel{(k)}{\sim} (c_1(y_1, w_1), \dots, c_n(y_n, z_n)) \equiv \underline{I}$$

Whenever  $c_i: R_+^2 \rightarrow R_+$  is right continuous nondecreasing function,  $i = 1, \dots, n$ .

The following lemma will be used in the proof of Theorem 2.4.

Lemma 2.5. Let  $\underline{X}$ ,  $\underline{Y}$ ,  $\underline{Z}$  and  $\underline{W}$  be as in Theorem 2.4. Then

$$(2.4) \quad E\psi(\underline{X}, \underline{Z}) < E\psi(\underline{Y}, \underline{W})$$

for all  $\psi$  such that

$$(2.5) \quad \phi_{\underline{Z}}(\cdot) \equiv \psi(\cdot, \underline{Z}) \in M_{k,n} \text{ for all } \underline{Z} > \underline{0},$$

$$(2.6) \quad \phi^{\underline{X}}(\cdot) \equiv \psi(\underline{X}, \cdot) \in M_{k,n} \text{ for all } \underline{X} > \underline{0},$$

provided the expectations in (2.4) exist.

Proof. Denote by  $F_{\underline{X}}$ ,  $F_{\underline{Y}}$ ,  $F_{\underline{Z}}$ ,  $F_{\underline{W}}$  the distributions of  $\underline{X}$ ,  $\underline{Y}$ ,  $\underline{Z}$ ,  $\underline{W}$ , respectively. Let  $\psi$  satisfy (2.5) and (2.6). Then

$$\begin{aligned} E\psi(\underline{X}, \underline{Z}) &= \int [\int \psi(\underline{x}, \underline{z}) dF_{\underline{Z}}(\underline{z})] dF_{\underline{X}}(\underline{x}) \\ &\stackrel{(2.6)}{<} \int [\int \psi(\underline{x}, \underline{w}) dF_{\underline{W}}(\underline{w})] dF_{\underline{X}}(\underline{x}) \\ &= \int [\int \psi(\underline{x}, \underline{w}) dF_{\underline{X}}(\underline{x})] dF_{\underline{W}}(\underline{w}) \\ &\stackrel{(2.5)}{<} \int [\int \psi(\underline{y}, \underline{w}) dF_{\underline{Y}}(\underline{y})] dF_{\underline{W}}(\underline{w}) \\ &= E\psi(\underline{Y}, \underline{W}), \end{aligned}$$

and (2.4) follows. ||

Proof of Theorem 2.4. Let  $\phi \in M_k$ . Consider the function  $\eta_{\underline{Z}}$  defined, for a fixed  $\underline{Z} > \underline{0}$ , by  $\eta_{\underline{Z}}(\cdot, \dots, \cdot) = \phi(c_1(\cdot, z_1), \dots, c_n(\cdot, z_n))$ . By Lemma 2.3,  $\eta_{\underline{Z}} \in M_k$  for all  $\underline{Z} > \underline{0}$ . Similarly the function  $\eta^{\underline{X}}$ , which is defined for each fixed  $\underline{X} > \underline{0}$ , by  $\eta^{\underline{X}}(\cdot, \dots, \cdot) = \phi(c_1(x_1, \cdot), \dots, c_n(x_n, \cdot))$ , is also a

member of  $M_k$  for all  $\underline{x} > \underline{0}$ . Hence, by Lemma 2.5,

$$\begin{aligned} E\phi(\underline{S}) &= E\phi(c_1(x_1, Z_1), \dots, c_n(x_n, Z_n)) \\ &\leq E\phi(c_1(Y_1, W_1), \dots, c_n(Y_n, W_n)) = E\phi(\underline{I}). \end{aligned}$$

This is true for every  $\phi \in M_k$ . Hence  $\underline{S} \stackrel{(k)}{\sim} \underline{I}$ . ||

Theorems 2.6 and 2.7 below are special cases of Theorem 2.4. Theorem 2.6 shows that the ordering  $\stackrel{(k)}{\sim}$  is preserved under convolutions.

Theorem 2.6. Let  $\underline{X}$ ,  $\underline{Y}$ ,  $\underline{Z}$  and  $\underline{W}$  be as in Theorem 2.4. Then

$$\underline{X} + \underline{Z} \stackrel{(k)}{\sim} \underline{Y} + \underline{W}.$$

One consequence of Theorem 2.6 is the following. If  $\underline{X}$ ,  $\underline{Y}$ , and  $\underline{Z}$  belongs to  $\mathfrak{X}_n$  such that  $\underline{X} \stackrel{(k)}{\sim} \underline{Y}$  and  $\underline{Z}$  is independent of  $\underline{X}$  and  $\underline{Y}$  then

$$\underline{X} + \underline{Z} \stackrel{(k)}{\sim} \underline{Y} + \underline{Z}.$$

Theorem 2.7. Let  $\underline{X}$ ,  $\underline{Y}$ ,  $\underline{Z}$  and  $\underline{W}$  be as in Theorem 2.4. Then

$$\begin{aligned} (\min(X_1, Z_1), \dots, \min(X_n, Z_n)) &\stackrel{(k)}{\sim} (\min(Y_1, W_1), \dots, \min(Y_n, W_n)), \\ (\max(X_1, Z_1), \dots, \max(X_n, Z_n)) &\stackrel{(k)}{\sim} (\max(Y_1, W_1), \dots, \max(Y_n, W_n)). \end{aligned}$$

In the next theorem,  $\underline{X}^{(i)}$  denotes  $(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$  and  $\underline{Y}^{(i)}$  denotes  $(Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_n)$ ,  $i = 1, \dots, n$ .

Theorem 2.8. Let  $\underline{X}$ ,  $\underline{Y} \in \mathfrak{X}_n$ . Suppose  $\underline{X} \stackrel{(k)}{\sim} \underline{Y}$ . (a) If  $1 < k < n$  then

$$(2.7) \quad \underline{X}^{(i)} \stackrel{(k-1)}{\sim} \underline{Y}^{(i)}.$$

(b) If  $\underline{X}$  and  $\underline{Y}$  have all their mass on  $\{\underline{x}: x_1 > 0, \dots, x_n > 0\}$ , [i.e., if none of the  $X_i$ 's or  $Y_i$ 's is zero with positive probability] and if

$$1 < k < n - 1, \text{ then}$$

$$(2.8) \quad \underline{X}^{(i)} \stackrel{(k)}{\sim} \underline{Y}^{(i)}.$$

Proof of (a). By definition,  $\underline{X} \stackrel{(k)}{\sim} \underline{Y}$  if and only if  $(a_1 X_1, \dots,$

$a_n x_n)_{(k)} \stackrel{\xi^t}{\sim} (a_1 y_1, \dots, a_n y_n)_{(k)}$  for all  $a_i > 0$ ,  $i = 1, \dots, n$ . Hence

$$(2.9) \quad \lim_{a_i \rightarrow 0} (a_1 x_1, \dots, a_n x_n)_{(k)} \stackrel{\xi^t}{\sim} \lim_{a_i \rightarrow 0} (a_1 y_1, \dots, a_n y_n)_{(k)}$$

for all  $a_j > 0$ ,  $j \neq i$ . But (2.9) is the same as

$$(a_1 x_1, \dots, a_{i-1} x_{i-1}, a_{i+1} x_{i+1}, \dots, a_n x_n)_{(k-1)} \stackrel{\xi^t}{\sim} (a_1 y_1, \dots, a_{i-1} y_{i-1}, a_{i+1} y_{i+1}, \dots, a_n y_n)_{(k-1)} \text{ for all } a_j > 0, j \neq i, \text{ and (2.7) follows.} ||$$

Proof of (b). Again  $\underline{x} \stackrel{(k)}{\sim} \underline{y}$  if and only if  $(a_1 x_1, \dots,$

$a_n y_n)_{(k)} \stackrel{\xi^t}{\sim} (a_1 y_1, \dots, a_n y_n)_{(k)}$  for all  $a_i > 0$ ,  $i = 1, \dots, n$ . Hence

$$(2.10) \quad \lim_{a_i \rightarrow \infty} (a_1 x_1, \dots, a_n x_n)_{(k)} \stackrel{\xi^t}{\sim} \lim_{a_i \rightarrow \infty} (a_1 y_1, \dots, a_n y_n)_{(k)}$$

for all  $a_j > 0$ ,  $j \neq i$ . Since  $\underline{x}$  and  $\underline{y}$  are positive with probability one, it follows that (2.10) is the same as

$$(a_1 x_1, \dots, a_{i-1} x_{i-1}, a_{i+1} x_{i+1}, \dots, a_n x_n)_{(k)} \stackrel{\xi^t}{\sim} (a_1 y_1, \dots, a_{i-1} y_{i-1}, a_{i+1} y_{i+1}, \dots, a_n y_n)_{(k)} \text{ for all } a_j > 0, j \neq i, \text{ and (2.8) follows.} ||$$

Theorem 2.8 says that if the  $n$ -dimensional vectors  $\underline{x}$  and  $\underline{y}$  satisfy  $\underline{x} \stackrel{(k)}{\sim} \underline{y}$  then (under the proper conditions) the  $(n-1)$ -dimensional marginals satisfy the orderings  $\stackrel{(k-1)}{\sim}$  and  $\stackrel{(k)}{\sim}$ . By induction, for proper choices of  $k$ , the  $(n-2)$ -dimensional marginals satisfy the orderings  $\stackrel{(k-2)}{\sim}$ ,  $\stackrel{(k-1)}{\sim}$  and  $\stackrel{(k)}{\sim}$ , and so on. The reader may wish to list all the orderings that follow, for example, for  $n = 6$ , from  $\stackrel{(4)}{\sim}$  for the  $m$ -dimensional marginals,  $m = 5, 4, 3, 2$ . In general, if  $\underline{x}$  and  $\underline{y}$  satisfy the conditions of Theorem 2.8 (a) and (b) then, for  $1 \leq m \leq n$ , any  $m$ -dimensional marginal of  $\underline{x}$  is  $\stackrel{(k)}{\sim}$  than the

respective  $m$ -dimensional marginal of  $\underline{Y}$  whenever  $\max(1, m+k-n) \leq l \leq \min(k, m)$ .

### 3. Examples and applications.

In this section some examples of functions  $\phi$  in  $M_k$  will be given and then an application in reliability theory will be discussed.

In general, every distribution function in  $M_1$  determines a function  $\phi$  in  $M_{k,n}$  as described in (2.1). In the following examples we describe in some detail some functions  $\phi$  in  $M_{2,3}$ .

Example 3.1. Let  $F_i$ ,  $i = 1, 2, 3$  and  $G$  be univariate probability distributions on  $R_+$  and let  $U_1, U_2, U_3$  and  $W$  be independent random variables with distributions  $F_1, F_2, F_3$  and  $G$  respectively. If  $V_i \equiv \max(U_i, W)$ ,  $i = 1, 2, 3$ , then the joint distribution of  $V_1, V_2$  and  $V_3$  is given by

$$f(v_1, v_2, v_3) = F_1(v_1)F_2(v_2)F_3(v_3)G(\min(v_1, v_2, v_3)), \quad v_i \geq 0, \quad i = 1, 2, 3.$$

Thus the function  $\phi$  defined by

$$\begin{aligned} \phi(v_1, v_2, v_3) &= f(v_1, v_2, \infty) + f(v_1, \infty, v_3) + f(\infty, v_2, v_3) - 2f(v_1, v_2, v_3) \\ &= F_1(v_1)F_2(v_2)G(\min(v_1, v_2)) \\ &\quad + F_1(v_1)F_3(v_3)G(\min(v_1, v_3)) \\ &\quad + F_2(v_2)F_3(v_3)G(\min(v_2, v_3)) \\ &\quad - 2F_1(v_1)F_2(v_2)F_3(v_3)G(\min(v_1, v_2, v_3)), \quad v_i \geq 0, \quad i = 1, 2, 3, \end{aligned}$$

is a member of  $M_{2,3}$ .

For example, if  $F_1, F_2, F_3$  and  $G$  are the uniform distributions on the interval  $[0, 1]$  then for  $0 \leq v_i \leq 1$ ,  $i = 1, 2, 3$ ,

$$\begin{aligned} \phi(v_1, v_2, v_3) &= v_1 v_2 \min(v_1, v_2) + v_1 v_3 \min(v_1, v_3) \\ &\quad + v_2 v_3 \min(v_2, v_3) - 2v_1 v_2 v_3 \min(v_1, v_2, v_3). \end{aligned}$$

Example 3.2. Let  $U_1, U_2, U_3, W$  and  $F_1, F_2, F_3, G$  be as in Example 3.1.

Denote  $\bar{F}_i = 1 - F_i$ ,  $\bar{G} = 1 - G$ . If  $V_i = \min(U_i, W)$ ,  $i = 1, 2, 3$ , then the joint survival function of  $V_1, V_2$  and  $V_3$  is given by

$$(3.1) \quad \begin{aligned} \bar{F}(v_1, v_2, v_3) &= P\{V_1 > v_1, V_2 > v_2, V_3 > v_3\} \\ &= \bar{F}_1(v_1)\bar{F}_2(v_2)\bar{F}_3(v_3)\bar{G}(\max(v_1, v_2, v_3)), \quad v_i > 0, i = 1, 2, 3. \end{aligned}$$

The joint distribution  $f$  of  $V_1, V_2$  and  $V_3$  can be obtained from  $\bar{F}$  using the formula

$$(3.2) \quad \begin{aligned} f(v_1, v_2, v_3) &= 1 - \bar{F}(v_1, 0, 0) - \bar{F}(0, v_2, 0) - \bar{F}(0, 0, v_3) \\ &\quad + \bar{F}(v_1, v_2, 0) + \bar{F}(v_1, 0, v_3) + \bar{F}(0, v_2, v_3) \\ &\quad - \bar{F}(v_1, v_2, v_3), \quad v_i > 0, i = 1, 2, 3. \end{aligned}$$

The function  $\phi$  defined by

$$(3.3) \quad \begin{aligned} \phi(v_1, v_2, v_3) &= f(v_1, v_2, \infty) + f(v_1, \infty, v_3) + f(\infty, v_2, v_3) \\ &\quad - 2f(v_1, v_2, v_3), \quad v_i > 0, i = 1, 2, 3, \end{aligned}$$

belongs to  $M_{2,3}$ .

Plugging (3.2) into (3.3) one obtains

$$(3.4) \quad \begin{aligned} \mu(v_1, v_2, v_3) &= 1 - \bar{F}(v_1, v_2, 0) - \bar{F}(v_1, 0, v_3) - \bar{F}(0, v_2, v_3) \\ &\quad + 2\bar{F}(v_1, v_2, v_3), \quad v_i > 0, i = 1, 2, 3. \end{aligned}$$

Choosing various univariate distribution functions  $F_i$ ,  $i = 1, 2, 3$ , and  $G$  in (3.1) and plugging the resulting  $\bar{F}$  in (3.4), one can obtain explicit expressions for various members of  $M_{2,3}$ .

For example, if  $U_i$ ,  $i = 1, 2, 3$ , and  $W$  are uniformly distributed on  $[0, 1]$  then for  $0 < v_i < 1$ ,  $i = 1, 2, 3$ ,

$$(3.5) \quad \begin{aligned} \phi(v_1, v_2, v_3) &= 1 - (1-v_1)(1-v_2)(1-\max(v_1, v_2)) \\ &\quad - (1-v_1)(1-v_3)(1-\max(v_1, v_3)) \\ &\quad + 2(1-v_1)(1-v_2)(1-v_3)(1-\max(v_1, v_2, v_3)). \end{aligned}$$

If  $V_i$ ,  $i = 1, 2, 3$ , and  $W$  are standard (mean one) exponential random variables [then  $(V_1, V_2, V_3)$  has a Marshall-Olkin (1967) multivariate exponential distribution] then  $G(t) = F_i(t) = 1 - e^{-t}$ ,  $t > 0$ ,  $i = 1, 2, 3$ . Plugging these in (3.1) and (3.4) one obtains the following member of  $M_{2,3}$ :

$$(3.6) \quad \phi(v_1, v_2, v_3) = 1 - e^{-v_1 - v_2 - \max(v_1, v_2)} - e^{-v_1 - v_3 - \max(v_1, v_3)} - e^{-v_2 - v_3 - \max(v_2, v_3)} + 2e^{-v_1 - v_2 - v_3 - \max(v_1, v_2, v_3)}, \quad v_i > 0, i = 1, 2, 3.$$

Of course, that  $\phi$  of (3.6) is a member of  $M_{2,3}$  can also follow at once from Theorems 2.1, 2.2 and the fact that  $\phi$  of (3.5) is in  $M_{2,3}$ .

Example 3.3. The function  $\bar{F}$  defined by

$$\bar{F}(v_1, v_2, v_3) = (1 + v_1 + v_2 + v_3)^{-1}, \quad v_i > 0, i = 1, 2, 3$$

is a survival function (see, e.g., Takahashi (1965)). Substituting it in (3.2) one obtains a distribution function  $f$ . Substituting this  $f$  in (3.4) one obtains  $\phi$ , a member of  $M_{2,3}$ , defined by

$$\phi(v_1, v_2, v_3) = 1 - (1 + v_1 + v_2)^{-1} - (1 + v_1 + v_3)^{-1} - (1 + v_2 + v_3)^{-1} + 2(1 + v_1 + v_2 + v_3)^{-1}, \quad v_i > 0, i = 1, 2, 3.$$

Application 3.4 (reliability theory). Every collection  $X_1, \dots, X_n$  of nonnegative random variables can be thought of as a collection of lifelengths of devices. For  $a_i > 0$ ,  $i = 1, \dots, n$ , the scaled lifelengths  $a_1 X_1, \dots, a_n X_n$  have been of interest in many studies (see, e.g., El-Newehi (1984), Marshall and Shaked (1985a,b) and references therein). Note that  $(a_1 X_1, \dots, a_n X_n)_{(k)}$  is the lifelength of an  $(n-k+1)$ -out-of- $n$  system with component



lifetimes  $a_1 X_1, \dots, a_n X_n$ .

Let  $\underline{X} = (X_1, \dots, X_n)$  and  $\underline{Y} = (Y_1, \dots, Y_n)$  be two vectors of random lifelengths with distributions  $F$  and  $G$  respectively. In some applications the condition  $\underline{X} \stackrel{(k)}{\sim} \underline{Y}$  naturally holds or is not very hard to prove (see below for more details). Then Theorem 2.1 yields a host of useful inequalities.

In order to verify  $\underline{X} \stackrel{(1)}{\sim} \underline{Y}$  [respectively,  $\underline{X} \stackrel{(n)}{\sim} \underline{Y}$ ], it follows from Propositions 1.1 and 1.2 that all that one has to do is to show  $F(\underline{t}) < G(\underline{t})$  [respectively  $F(\underline{t}) > G(\underline{t})$ ],  $\underline{t} > \underline{0}$ . In order to verify  $\stackrel{(k)}{\sim}$  for  $1 < k < n$ , one just has to show (see the proof of Theorem 2.1)

$$(3.7) \quad \sum_{m=n-k+1}^n (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I \in \psi_m} F(\underline{t}_I, 0), \underline{t} > 0, \\ < \sum_{m=n-k+1}^n (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I \in \psi_m} G(\underline{t}_I, 0), \underline{t} > 0,$$

or, equivalently,

$$(3.8) \quad \sum_{m=k}^n (-1)^{m-k} \binom{m-1}{k-1} \sum_{I \in \psi_m} F(\underline{t}_I, \infty) > \sum_{m=k}^n (-1)^{m-k} \binom{m-1}{k-1} \sum_{I \in \psi_m} G(\underline{t}_I, \infty), \underline{t} > 0.$$

Clearly the ordering  $\stackrel{st}{\sim}$  implies  $\stackrel{(k)}{\sim}$  (here, for every two random vectors  $\underline{X}$  and  $\underline{Y}$ , the notation  $\underline{X} \stackrel{st}{\sim} \underline{Y}$  means

$$(3.9) \quad \text{Eg}(\underline{X}) \leq \text{Eg}(\underline{Y})$$

for all measurable nondecreasing functions  $g$  for which the expectations exist). Note however that (3.9) is a much stronger requirement than (3.7) or (3.8). Hence (3.9) may not hold when (3.7) or (3.8) hold. Even if (unknown to the researcher) (3.9) holds it still may be possible only to show (3.7) or (3.8). Thus the advantage of  $\stackrel{(k)}{\sim}$  over  $\stackrel{st}{\sim}$  is in the relative simplicity of its verification and the fact that it still implies many useful inequalities.

Since the ordering  $\stackrel{st}{\sim}$  implies  $\stackrel{(k)}{\sim}$ , it follows that the

inequalities of Theorem 2.1 apply in many applications in reliability theory and elsewhere. For example, Block, Savits and Shaked (1985) give conditions under which a nonnegative random vector  $\underline{T} = (T_1, \dots, T_n)$  satisfies

$$(3.10) \quad [(T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n) | T_i = t'] \\ \leq^t [(T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n) | T_i = t]$$

whenever  $0 < t < t'$ ,  $i = 1, \dots, n$ . Denoting the left hand side of (3.10) by  $\underline{X}$  and the right hand side of (3.10) by  $\underline{Y}$  (here  $\underline{X}$  and  $\underline{Y}$  are  $(n-1)$ -dimensional) it follows that  $\underline{X} \stackrel{(k)}{\sim} \underline{Y}$  and the inequalities described in Theorem 2.1 apply.

Application 3.5 (systems with spare parts). Consider two systems of  $n$  components. Denote the lifetimes of the components of the first system by  $X_1, \dots, X_n$  and of the second system by  $Y_1, \dots, Y_n$ . Suppose each component in each system has a cold spare which starts to live upon the failure of the component. Denote the lifetimes of the spares by  $Z_1, \dots, Z_n$  and  $W_1, \dots, W_n$  where  $Z_i$  [respectively,  $W_i$ ] is the lifetimes of the spare which replaces the component with lifetime  $X_i$  [respectively,  $Y_i$ ],  $i = 1, \dots, n$ . The  $X_i$ 's [respectively,  $Y_i$ 's,  $Z_i$ 's,  $W_i$ 's] among themselves may be dependent, but we assume that  $\underline{X}$  is independent of  $\underline{Z}$  and  $\underline{Y}$  is independent of  $\underline{W}$ .

The lifetime  $T_1$  of the first system then is determined by  $\underline{X} + \underline{Z}$ , say

$$(3.11) \quad T_1 = \tau(\underline{X} + \underline{Z})$$

where  $\tau$  is a coherent life function in the sense of Esary and Marshall (1970). Suppose that the lifetime  $T_2$  of the second system is

$$(3.12) \quad T_2 = \tau(\underline{Y} + \underline{W})$$

where the  $\tau$  in (3.12) is the same as the  $\tau$  in (3.11).

If  $\underline{X} \stackrel{(k)}{\sim} \underline{Y}$  and  $\underline{Z} \stackrel{(k)}{\sim} \underline{W}$  then, by Theorem 2.6,  $\underline{X} + \underline{Z} \stackrel{(k)}{\sim} \underline{Y} + \underline{W}$ . Thus

various inequalities regarding  $T_1$  and  $T_2$  can be obtained from Theorem 2.1. For example, if  $\tau$  is the coherent life function which corresponds to an  $(n-k+1)$ -out-of- $n$  system then we have

$$(3.13) \quad T_1 \leq^t T_2.$$

If the spares are warm standbys then, using the above notation, the lifetime  $T_1$  of the first system is determined by  $(\max(X_1, Z_1), \dots, \max(X_n, Z_n))$ , say

$$\tilde{T}_1 = \tau(\max(X_1, Z_1), \dots, \max(X_n, Z_n))$$

and the lifetime  $T_2$  of the second system is

$$\tilde{T}_2 = \tau(\max(Y_1, W_1), \dots, \max(Y_n, W_n)).$$

Using Theorem 2.7 we get

$$\tilde{T}_1 \leq^t \tilde{T}_2$$

when  $\tau$  is as described before (3.13).

As another example, suppose  $n-1$  of the components with lifetimes  $X_i$  [respectively  $Y_i$ ] are used, with the corresponding spare parts, for an  $(n-k)$ -out-of- $(n-1)$  system. Denote the lifetime of the resulting system by  $S_1$  [respectively,  $S_2$ ]. Then, from Theorem 2.8 it follows that

$$S_1 \leq^t S_2.$$

Similar stochastic ordering applies to systems which use "second-hand" components as described in Block, Bueno, Savits and Shaked (1984). We omit the details.

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